Dynamical Analysis of Predator-Prey Model Leslie-Gower with Omnivore

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Abstract

This article discussed a dynamical analysis on a model of predator-prey Leslie-Gower with omnivore. The dynamical analysis was done by determining the equilibrium point with its existing condition and analyzing the local stability of the equilibrium point. Based on the analysis, there are seven points of equilibrium. Three of them always exist while the four others exist under certain conditions. Four points of equilibrium, which are unstable, while the other three equilibrium point are locally asymptotically stable under certain conditions. Moreover, numerical simulations were also conducted to illustrate the analytics. The results of numerical simulations agree with the results of the dynamical analysis.

Keywords: local stability, omnivore, predator-prey models, the equilibrium point.

INTRODUCTION

Lotka-Volterra model was firstly introduced Lotka in 1925 and Volterra in 1926 [1]. Lotka-Volterra’s study has produced a simple model of predation or interaction between two species in an ecosystem. They also have introduced classical Lotka-Volterra model, which is currently developed by researchers [2].

In 1948, Leslie discussed Lotka-Volterra model and found impossibility in a model, which is infinity in predator growth [3]. Therefore, Leslie and Gower introduced a new name’s predator-prey model, which is modification of Lotka-Volterra’s model. The model is known as Leslie-Gower Predator-Prey Model. Leslie-Gower have modified Lotka-Volterra’s predator-prey model by assuming that the predation of predator is limited, which means that the predation of predator will not more carrying capacity of prey. Leslie-Gower two dimension models as:

\[
\frac{dx}{dt} = (r_1 - a_1 y - b_1 x)x, \\
\frac{dy}{dt} = (r_2 - a_2 y - b_2 x)y.
\]  

(1)

with \(x(t)\) state the population density of prey and \(y(t)\) state the population density of predator.

In 2015, Andayani and Kusumawinahyu [4] a three species predator – prey model, the third species are omnivores. This model is constructed by assuming there are just three species in such an ecosystem. The first species, called as prey (rice plant), the prey for the second and the third species. The second species, called as predator (carcass), only feeds on the first species and can extinct with prey. The third species, namely omnivores (mouse), eat the prey and carcasses of predator. Consequently, omnivores of predator only reduces the prey population but does not affect the predator growth. Assumed that the prey population grow logistically and any competition between omnivores [5]. Based on these assumption, the mathematical model representing those growth density of population rates by nonlinear ordinary differential equation system, namely

\[
\frac{dx}{dt} = x(1 - y - z - bx), \\
\frac{dy}{dt} = y(-c + x) \\
\frac{dz}{dt} = z(-e + fx + gy - bz).
\]  

(2)

In this model, \(x(t), y(t)\) and \(z(t)\) the density of prey, predator, and omnivore populations, respectively. All parameter of model (2) are positive. The death rates of the predator and omnivore are denoted by \(c\) and \(e\), respectively. The parameter \(f\) rivalry toward prey that effect increases of omnivore population, while the parameter \(g\) rivalry toward prey that effect increases of omnivore population. Parameter \(b\) and \(\beta\) carrying the capacity of the prey and omnivore, respectively [5]. The aim of this study is a dynamical analysis on a model of predator-prey...
Predator-Prey Model Leslie-Gower with Omnivore (Exviani et al)

Leslie-Gower with omnivores which is modified by Lotka-Volterra model with omnivore.

MATERIAL AND METHOD

In this study, predator-prey model by Leslie-Gower with omnivore. This model is constructed by assuming the third species are omnivores. This model is constructed by assuming there are just three species in such an ecosystem. The first species, called as prey (rice plant), the prey for the second and the third species. The second species, called as predator (carrion), only feeds on the first species and can extinct with prey. The third species, namely omnivores (mouse), eat the prey and carcasses of predator. Consequently, omnivores of predator only reduces the prey population but does not affect the predator growth. Assumed that the prey population grow logistically and any competition between omnivores.

Literature Study

Literature study related to the research process, such as the literature discussing the Leslie-Gower model, Lotka-Volterra model, omnivore, and forward-backward sweep method [7-12]. We also used other supporting references in problem solving in this study. In the Lotka-Volterra predator-prey model with omnivore, namely:

\[
\begin{align*}
\frac{dx}{dt} &= x(1 - y - z - bx), \\
\frac{dy}{dt} &= y(-c + x), \\
\frac{dz}{dt} &= z(-e + fx + gy - \beta z).
\end{align*}
\]

on this system has only five equilibrium point’s, namely:

\[
E_0 = (0,0,0) \\
E_1 = \left( \frac{1}{b}, 0, 0 \right) \\
E_2 = (c, 1 - bc, 0) \\
E_3 = \left( \frac{e + \beta}{f + \beta}, 0, \frac{f - b e}{g + \beta} \right) \\
E_4 = \left( \frac{b(1 - bc) - (f c - e)}{g + \beta}, g(1 - bc) + f(f c - e) \right).
\]

To accommodate biological meaning, the existence conditions for the equilibria require that they are nonnegative. It is obvious that \(E_0\) dan \(E_1\) always exist, \(E_2\) exist if \(bc < 1\), \(E_3\) exist if \(f > be\) and \(E_4\) the densities of omnivores and predators \(1 - bc\) has to be positive. Then, \(E_4\) exist if \(bc < 1\) and 0 < \(fc - e < \beta(1 - bc)\).

While, predator-prey model by Leslie-Gower with omnivore has seven equilibrium points (Table 1). So the predator-prey model by Leslie-Gower model with omnivores is more concrete in this case.

<table>
<thead>
<tr>
<th>Equilibrium Points</th>
<th>Existence Requirement</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E_1 = (0,0,0))</td>
<td>-</td>
</tr>
<tr>
<td>(E_2 = \left( \frac{kr_2}{a_2}, 0 \right))</td>
<td>-</td>
</tr>
<tr>
<td>(E_3 = \left( \frac{1}{b_1}, 0, 0 \right))</td>
<td>-</td>
</tr>
<tr>
<td>(E_4 = (0, y_4', z_4))</td>
<td>(c_3 k r_2 &gt; a_1 c_1)</td>
</tr>
<tr>
<td>(E_5 = (x_5, 0, z_5))</td>
<td>(r_1 c_2 &gt; b_1 c_1)</td>
</tr>
<tr>
<td>(E_6 = (x_6', y_6', 0))</td>
<td>(a_2 r_1 &gt; b_2 k r_2)</td>
</tr>
<tr>
<td>(E_7 = (x_7', y_7', z_7))</td>
<td>(c_4 r_1' + c_4 b_3 &gt; \frac{c_1}{c_2} (b_2 c_4 + b_3 c_3))</td>
</tr>
</tbody>
</table>

MATHEMATICAL MODEL

This study constructs Lotka-Volterra’s predator-prey model with omnivore (2). This model is developed by modify the predator that previously used Lotka-Volterra’s form to Leslie-Gower’s, which was examined by Leslie-Gower. This is based on the fact that predator depends on the available number of prey to establish. Therefore, the model is stated to be in the following equation system (3):

\[
\begin{align*}
\frac{dx}{dt} &= x(r_1 - b_1 x - b_2 y - b_3 z), \\
\frac{dy}{dt} &= y\left( \frac{-a_2}{x + k} \right), \\
\frac{dz}{dt} &= z(-c_1 + c_2 x + c_3 y - c_4 z).
\end{align*}
\]

with \(x = x(t)\), \(y = y(t)\), and \(z = z(t)\) state the population density of prey, predator, and omnivore. All of the parameters are positive in value. Parameters \(r_1\) and \(r_2\) respectively show intrinsic growth of prey and predator. \(b_1\) is the coefficient of competition between prey, \(b_2\) is the predator interaction coefficient between predator to prey and \(b_3\) is a predatory interaction between omnivores against prey. Whereas \(a_2\) is an interaction parameter between predators and parameter \(k\) is a parameter of protection against
predators. Parameter $c_1$ is omnivorous natural death, $c_2$ partially omnivorous predictor coefficient to prey, $c_3$ as predator coefficient of predator carcass, $c_4$ is competition between omnivorous population. While, tribal form $(x+k)$ can be interpreted as scarcity of prey may stimulate predators to replace foot sources with other alternatives. Therefore, it is assumed that predators depend not only on prey, but predators can eat other than prey in the prey environment. So in this article it is modeled by adding a positive constant $k$ to the division.

RESULT AND DISCUSSION

All parameters of Equation (3.1) in this study are assumed positive in value. Parameters $r_1$ and $r_2$ consecutively show intrinsic growth of prey and predator. $b_1$ is the competition coefficient among preys, $b_2$ is the predation interaction coefficient between predator and prey, and $b_3$ is the predation interaction between omnivore and prey. Meanwhile, $a_2$ is the interaction parameter among predators, and parameter $k$ is the protection parameter against predator. Parameter $c_1$ is the natural death of omnivore, $c_2$ is the predation coefficient of omnivore on prey, $c_3$ is the predation coefficient on the carcass of predator, and $c_4$ is the competition among omnivore populations.

Equilibrium Point and Existence

The point of Equilibrium (3) is solution for sytem:

The system has seven points of equilibrium, namely $E_1 = (0,0,0)$, $E_2 = (0, \frac{kr_3}{a_3}, 0)$, and $E_2 = (\frac{x_1}{a_1}, 0,0)$ in which the three points exist unconditionally, equilibrium point $E_3 = (0, \frac{kr_3}{a_3}, 0, \frac{r_3}{a_3})$ exists with the condition of $\frac{c_1}{a_1} > \frac{1}{2}$, equilibrium point $E_4 = (\frac{x_2}{a_2}, \frac{r_3}{a_3}, \frac{r_3}{a_3}, 0)$ exists if $z = \frac{c_1}{a_1} > \frac{1}{2}$, equilibrium point $E_5 = (\frac{x_3}{a_3}, 0, 0)$ exists if $a_2 \frac{x_2}{a_2} > c_3$, equilibrium point $E_6 = (\frac{x_4}{a_4}, 0, 0)$ exists if $a_2 \frac{x_4}{a_4} > c_3$.

Stability Analysis

The local stability of system (3) for each equilibrium point is as follows.

a) $E_1 = (0,0,0)$ is unstable

b) $E_2 = \left(0, \frac{kr_3}{a_3}, 0\right)$ is stable if $r_1 a_2 < b_2 r_2 k$ and $c_2 r_3 \frac{k}{a_3} > c_1 a_2$

c) $E_3 = \left(\frac{x_1}{a_1}, 0,0\right)$ is unstable

d) $E_4 = \left(0, \frac{kr_3}{a_3}, 0, \frac{r_3}{a_3}\right)$ is stable if $c_4 r_1 \frac{1}{a_1} + c_1 b_2 < \frac{k r_3}{a_3}$

e) $E_5 = \left(0, 0, 0\right)$ is unstable

This stability analysis uses the criteria of Routh-Hurwitz $\lambda^3 + A\lambda^2 + B\lambda + C = 0$. The characteristic equation will have negative roots if and only if $A > 0, C > 0$ and $AB > C$. Therefore, it can be concluded that $AB - C > 0$. If the condition is met, the equilibrium point of $E_7$ will be stable.

Proof

i. Jacobi matrix system (3) for $E_1$ is

$$J(E_1) = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & -c_1 \end{bmatrix}$$

The three Eigen values of the matrix $J(E_1)$ are positive, so $E_1$ is unstable.

ii. Jacobi matrix $E_2 = \left(0, \frac{kr_3}{a_3}, 0\right)$ is

$$J = \begin{bmatrix} r_1 - b_2 \frac{r_3}{a_2} & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & -c_1 + \frac{c_2 \frac{r_3}{a_3} k}{a_2} \end{bmatrix}$$

which has eigen values

$$\lambda_1 = \frac{r_1 - b_2 \frac{r_3}{a_2}}{a_2}, \lambda_2 = -r_2, \lambda_3 = -c_1 + \frac{c_2 \frac{r_3}{a_3} k}{a_2}$$
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The equilibrium point \((E_4)\) stable, if 
\[ r_1 a_2 < b_2 r_1 k \text{ and } c_2 r_2 k > c_1 a_2. \]

iii. Jacobi matrix on equilibrium point 
\[ E_3 = \left( \frac{r_1}{a_2}, 0, 0 \right) \]

\[
J = \begin{bmatrix}
-r_1 & \frac{r_1 b_2}{b_1} & -\frac{r_1 b_2}{b_1} \\
0 & -\frac{r_2}{b_2} & 0 \\
0 & 0 & -c_1 + \frac{c_2 r_1}{b_1}
\end{bmatrix}
\]

has eigen values 
\[ \lambda_1 = -r_1, \lambda_2 = r_2, \lambda_3 = -c_1 + \frac{c_2 r_1}{b_1} \] so 
\(E_3\) unstable.

iv. Jacobi matrix on equilibrium point 
\[ E_4 = \left( \frac{r_1 a_2}{a_2}, \frac{c_2 r_2 - c_1 a_2}{a_2}, 0 \right) \]

\[
J = \begin{bmatrix}
\frac{r_1}{a_2} - \frac{b_2 y_4}{b_2 z_4} & 0 & 0 \\
0 & -\frac{r_2}{a_2} & 0 \\
0 & 0 & -c_1 + c_3 z_4 - c_1 - 2 c_4 z_4 + c_1 z_4
\end{bmatrix}
\]

because 
\[ x = 0, r_1 y_2 + y_4 = 0, \text{ and } -c_1 + c_3 x_4 + c_2 y_4 + c_4 z_4 = 0 \text{ so } E_4 \text{ stable if }\]
\[ \lambda_1 = \frac{r_1}{a_2} - \frac{b_2 y_4}{b_2 z_4} - \frac{b_2 y_4}{b_2 z_4} < 0, \]
\[ \frac{r_1 a_2 c_4 - b_2 r_2 c_4 - b_2 r_2 c_3 + b_2 c_2 c_1}{c_2 z_4} < 0, \]
\[ c_4 n_1 + c_1 b_1 < \frac{k_r}{a_2} (b_2 c_4 + b_1 c_1). \]

v. Jacobi matrix on the equilibrium point 
\[ E_5 = \left( \frac{r_1 c_2 + c_2 c_1}{c_2 b_1 + c_2 b_1}, 0, c_1 c_1 - c_1 b_1 \right) \]

\[
J(E_5) = \begin{bmatrix}
\frac{r_1}{a_2} - 2 b_2 x_5 - b_2 z_5 & -b_2 x & -b_2 x \\
0 & c_2 z_5 & c_2 z_5 - c_1 - 2 c_4 z_5 + c_1 z_5 \\
0 & c_2 z_5 & c_2 z_5 - c_1 - 2 c_4 z_5 + c_1 z_5
\end{bmatrix}
\]

with, 
\[ r_1 - b_1 x - b_2 y - b_3 z = 0 \Rightarrow r_1 - b_1 x_5 - b_3 z_5 = 0, \]
\[ y = 0, \]
\[ -c_1 + c_3 x_4 + c_2 y_4 + c_4 z_4 = 0 \Rightarrow -c_1 + c_3 x_4 + c_4 z_4 = 0, \]
so 
\[ J(E_5) = \begin{bmatrix}
-b_1 x_5 & -b_2 x & -b_2 x \\
0 & c_2 z_5 & c_2 z_5 - c_1 - 2 c_4 z_5 + c_1 z_5 \\
0 & c_2 z_5 & c_2 z_5 - c_1 - 2 c_4 z_5 + c_1 z_5
\end{bmatrix} \]

\[ |J(E_5) - \lambda I| = \begin{vmatrix}
\lambda - r_1 & -b_1 x_5 & -b_2 x \\
0 & c_2 z_5 & c_2 z_5 - c_1 - 2 c_4 z_5 + c_1 z_5 \\
0 & c_2 z_5 & c_2 z_5 - c_1 - 2 c_4 z_5 + c_1 z_5
\end{vmatrix}\]

\[ = (\lambda - r_1) \begin{vmatrix}
\lambda - b_2 x_5 & -b_2 x \\
0 & c_2 z_5 & c_2 z_5 - c_1 - 2 c_4 z_5 + c_1 z_5
\end{vmatrix}\]

\[ = 0. \]

vi. Jacobi matrix on the equilibrium point 
\[ E_6 = \left( \frac{r_1 c_2 - b_2 r_1 k_2 + r_2 (r_1 + b_2 r_1 k_2)}{r_2}, 0, 0 \right) \]

\[
J(E_6) = \begin{bmatrix}
\frac{r_1}{a_2} - b_1 x & -b_2 y & -b_2 x \\
0 & -c_1 + c_3 x_4 + c_4 y & 0 \\
0 & 0 & -c_1 + c_3 x_4 + c_4 y
\end{bmatrix}
\]

with, 
\[ r_1 - b_1 x - b_2 y - b_3 z = 0 \Rightarrow r_1 - b_1 x_5 - b_3 z_5 = 0, \]
\[ r_2 x + k y = 0 \Rightarrow y = \frac{r_2}{a_2} \left( \frac{y}{x + k} \right)^2 = \frac{r_2}{a_2} x \]
\[ z = 0, \]
so 
\[ J(E_6) = \begin{bmatrix}
\frac{r_1}{a_2} & -b_2 y & -b_2 x \\
0 & -c_1 + c_3 x_4 + c_4 y & 0 \\
0 & 0 & -c_1 + c_3 x_4 + c_4 y
\end{bmatrix} \]

result \( \lambda_1 < 0 \), with used characteric as follow \( \lambda_1, \lambda_2 > 0 \text{ and } \lambda_1 + \lambda_2 < 0 \) proven that \( E_6 \) unstable.
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\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} &= \begin{pmatrix} -\frac{r_2}{a_2} x(t) - b_2 x(t) - b_1 x(t) \\ -c_2 z(t) - c_1 z(t) - \lambda \\ c_2 z(t) - c_1 z(t) - \lambda \end{pmatrix} + \begin{pmatrix} (-r_2 - \lambda) \\ (r_2 + \lambda) \end{pmatrix} \\
\lambda^3 + A\lambda^2 + B\lambda + C &= 0,
\end{align*}
\]

With declaration as:

\[
A = c_2 z(t) + b_1 x(t) + r_1 > 0,
\]

\[
B = b_1 c_1 x(t) + b_2 c_2 x(t) + b_1 x(t) + c_4 r_2 x(t) + a_2 r_2^2 x(t) > 0,
\]

\[
C = \frac{a_1 b_1 c_1 r_2 x(t) + a_2 b_2 c_2 x(t) + b_1 c_1 r_2^2 x(t) + b_1 c_1 r_2^2 x(t)}{a_2},
\]

Based on criteria Routh-Hurwitz, the equilibrium point \( E_2 \) local asymptotically stable if \( A > 0 \), \( C > 0 \) and \( AB - C > 0 \).

Numerical Simulation

Numerical simulations are performed to see the validity of numerical analysis using the fourth-order Runge-Kutta method to illustrate the results of the analysis. There are several cases that are simulated in the discussion of this study, as follows.

Simulation I

Simulation I (Fig. 1) show \( E_2 \) exists, and conditions of stable \( E_2 \) are \( r_1 a_2 < b_2 kr_2 \) and \( c_1 a_2 > c_3 kr_2 \). Parameter being used \( r_1 = 0.5 \), \( b_1 = 0.05 \), \( b_2 = 0.6 \), \( r_2 = 0.1 \), \( r_2 = 0.7 \), \( a_1 = 0.2 \), \( k = 0.3 \), \( c_1 = 0.9 \), \( c_2 = 0.25 \), \( c_3 = 0.2 \), \( c_4 = 0.2 \). Thus, the equilibrium points of \( E_1 \) exists, \( E_2 \) exist, \( E_3 \) exists, and \( E_5 \) exists. The numerical simulation to equilibrium point \( E_5 = (0.105, 0) \), which is relevant to the analysis result which states that equilibrium \( E_2 \) is stable.

Simulation II

Simulation II, the stability conditions of \( E_2 \) are changed into \( r_1 a_2 > b_2 kr_2 \) and \( c_1 a_2 < c_3 kr_2 \). Parameter \( r_1 = 0.8 \), \( b_2 = 0.4 \), \( b_1 = 0.1 \), \( r_2 = 0.7 \), \( a_2 = 0.2 \), \( k = 0.3 \), \( c_1 = 0.3 \), \( c_2 = 0.25 \), \( c_3 = 0.6 \), \( c_4 = 0.2 \). Thus, it produces \( E_4 \) exists, \( E_5 \) exists, \( E_6 \) exists. Then, \( E_7 \) exists and is stable toward equilibrium point, so the initial value shows that \( E_7 \) is stable.
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**Simulation III**

The stability conditions in simulation III of $E_3$ are changed into $r_1a_2 > b_2kr_2$ dan $c_1a_2 > c_3kr_2$. Parameter $r_1 = 0.8$, $b_2 = 0.4$, $b_3 = 0.1$, $b_4 = 0.05$, $r_2 = 0.7$, $a_2 = 0.2$, $k = 0.3$, $c_1 = 0.9$, $c_2 = 0.25$, $c_3 = 0.2$, $c_4 = 0.2$. Thus, it produces $E_1$ exists, $E_2$ exists, $E_3$ exists, $E_5$ exists, and $E_6$ exists, but, in this case, it does not go to any point, so it exists but is unstable.

**Simulation IV**

The stability conditions in simulation IV, of $E_3$ are changed into $r_1a_2 < b_2kr_2$ dan $c_1a_2 < c_3kr_2$. Parameter $r_1 = 0.5$, $b_2 = 0.6$, $b_3 = 0.1$, $b_4 = 0.05$, $r_2 = 0.7$, $a_2 = 0.2$, $k = 0.3$, $c_1 = 0.3$, $c_2 = 0.25$, $c_3 = 0.6$, $c_4 = 0.2$. Thus, it produces $E_1$ exists, $E_2$ exists, $E_3$ exists, $E_4$ exists, and $E_5$ exists and unstable.

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Figure 2. Portrait Phase System Equation 3.1 for Simulation II

Figure 3. Portrait Phase System Equation 3.1 for Simulation III

Figure 4. Portrait Phase System Equation 3.1 for Simulation IV
Simulation V: the fifth simulation is conducted to show the stability characteristics of equilibrium point $E_4$. Based on the existence condition and the stability of equilibrium point $E_4$, parameter $c_4 r_1 + c_1 b_3 < \frac{r_1 k}{c_2} (b_2 c_4 + b_1 c_5)$, thus, it produces $E_1$ exists, $E_2$ exists, $E_3$ exists, $E_4$ exists, and $E_5$ exists towards equilibrium point $E_4$. This is relevant to the analysis result which states that the equilibrium point is stable.

**CONCLUSION**

The conclusions that are drawn based on the discussion of the thesis are as follows. Predator-prey model by Leslie-Gower with omnivore is obtained in a form of common differential equation system. There are seven equilibrium points in the model, there are three of them, i.e. $E_1$, $E_2$, and $E_3$, unconditionally exist and the other four, i.e. $E_4$, $E_5$, $E_6$, and $E_7$, conditionally exist. Of the seven equilibrium points, three of them, $E_1$, $E_2$, and $E_7$, have stability condition.

**REFERENCES**


