Numerical Solution of a Fractional-Order Predator-Prey Model with Prey Refuge and Additional Food for Predator

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Abstract

In this paper, a fractional-order predator-prey model with prey refuge and additional food for predator is solved numerically. For that aim, the model is discretized using a piecewise constant arguments. The equilibrium points of the discrete fractional-order model are investigated. Numerical simulations are conducted to see the stability of each equilibrium point. The numerical simulations show that stability of the equilibrium points is dependent on the time step.

Keywords: Additional Food, Fractional-Order, Predator-Prey, Prey Refuge.

INTRODUCTION

In ecology, understanding the dynamical relationship between prey and predator are center of goal [1]. Relationship between prey and predator plays important role for universal existence [2]. In 1928, Lotka [3] and Volterra [4] studied the relationship between prey and predator then introduced a predator-prey model. Recently, Ghosh et al. introduced the following prey-predator model with prey refuge and additional food for predator [5].

$$\begin{aligned}\frac{dx}{dt} &= x\left(1-\frac{x}{\gamma}\right)-\frac{(1-c')xy}{1+\alpha\xi+x},\\ \frac{dy}{dt} &= \frac{\beta((1-c')x+\xi)y}{1+\alpha\xi+x}-\delta y,\end{aligned} \tag{1}$$

where x and y denote the density of prey and predator with all parameters are positive. Model (1) explain that prey population grows logistically with environmental carrying capacity γ. In the second part of right-hand-side the prey population, parameter xy characterize as predation rate on prey by modified functional respon *Holling-type* II. The refuge on prey is exhibit in order to avoid the extinction of prey population caused by predation. In present, It is found that refuge shown by barnacle *Balanus glandula* (prey) in the higher intertidal stabilizes its interaction with snails *Thais* (the predator) [6]. Term c' be the prey refuge where $c' \in (0,1)$. Considering the limited resources caused by prey refuge, the predators need alternative food to prolong their survivability or it is known as additional food. For example in real life, in the Channel Island (archipelago off the French coast), golden eagle *Aquila chrysaetos* is reduced three resident fox populations by over 95%. The review reported that the *A. chrysaetos* are primarily sustained by hyper abundant alternative food species, the fox itself [7]. In model (1), the additional food is characterized as $\alpha\xi$ with both are quality and quantity of additional food, respectively. Parameter β represent the effect of predation, while δ be the death rate of predator. Solutions of model (1) has to satisfy the positivity condition that is $x > 0$ and $y > 0$ for all $t > 0$, due to biological nature.

Ghosh et al. showed that model (1) has three equilibrium points [5]: trivial equilibrium point $(0,0)$, axial equilibrium point $(\gamma,0)$ and co-existing equilibrium point (x^*,y^*) with $x^* = \frac{\delta+\alpha\delta\xi-\beta\xi}{\beta(1-c')-\delta}$ and $y^* = \left(1-\frac{x^*}{\gamma}\right)\left(\frac{1+\alpha\xi+x^*}{1-c'}\right)$.

Systematically, predator-prey model forms a system of first-order differential equations. In recent years, differential equation model with fractional-order atrracted researchers because of its applications in a lot of science areas [8]. Fractional-order differential equation provides excellent instrument for the description of memory and heredity of various materials and processes [9]. As a generalization of integer-order differential equation, the fractional-order exhibit dynamic behaviors, such as period-2^n-orbits [10]. Thus, fractional-order model gives interpretation of real phenomena realistically. Furthermore, the fractional-order model are naturally related to models with memory which exists in most biological models [11,12,13].

The fractional derivatives have several definitions. One of the most common definitions is the Caputo fractional derivatives which is often used in real applications.

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$$ {}_a^C D_t^n u(t) = \frac{1}{\Gamma(m-n)} \int_a^t \frac{u^m(\tau)}{(t-\tau)^{n-m+1}} d\tau \qquad (2)$$

where $(m-1) < n < m$ [14].

Population dynamics model becomes more appropriate and realistic when it is modeled by difference equations. A discretization of fractional-order models is needed because nonlinear fractional-order models generally have no analytical solution expressible in terms of a finite representation of elementary functions [15]. A discrete fractional-order is a something new for scientist. Miller and Ross pioneering this work [16]. The discrete memory effect of the model indicates that the momentum x_p depend on the past information $x_0, \dots, x_{p-1}$ [17].

In this paper, a fractional-order predator-prey model is discussed. Model (1) is then modificated by forming it into a discrete fractional-order with piecewise constant arguments. The aim of this study is to give an overview of population densities based on numerical simulations of the discrete fractional-order model.

MATERIAL AND METHOD

Discrete Model

Model (1) is modified into a fractional-order by replacing the first order derivative $\frac{dx}{dt}$ and $\frac{dy}{dt}$ into fractional order derivative $D_t^n x$ and $D_t^n y$. The modified model (1) is of the form

$$\begin{aligned} D_t^n x &= x(t)\left(1-\frac{x(t)}{\gamma}\right) - \frac{(1-c')x(t)y(t)}{1+\alpha\xi+x(t)}, \\ D_t^n y &= \frac{\beta\left((1-c')x(t)+\xi\right)y(t)}{1+\alpha\xi+x(t)} - \delta y(t), \end{aligned} \qquad (3)$$

where $0 < n < 1$. Following Elsadany and Matouk [11], and El-Sayed and Salman [18], the discretization process of model (3) can be performed to obtain

$$\begin{aligned} x_{m+1} &= x_m + \frac{s^n}{n\Gamma(n)}\left(x_m\left(1-\frac{x_m}{\gamma}\right)\right) - \\ &\quad \frac{s^n}{n\Gamma(n)}\left(\frac{(1-c')x_m y_m}{1+\alpha\xi+x_m}\right), \\ y_{m+1} &= y_m - \frac{s^n}{n\Gamma(n)}(\delta y_m) + \\ &\quad + \frac{s^n}{n\Gamma(n)}\left(\frac{\beta\left((1-c')x_m+\xi\right)y_m}{1+\alpha\xi+x_m}\right). \end{aligned} \qquad (4)$$

Notice that if $n \to 1$ then model (4) becomes the Euler discretization of model (1).

Determination of the Equilibrium Point

The equilibrium point of the model is obtained when the population growth rate are zero. The equilibrium point illustrates a constant solution of the model.

Numerical Simulation

Numerical simulations will be performed to study the behavior of model (4). Furthemore, the interpretation of the resulted numerical simulations is then discussed.

RESULT AND DISCUSSION

Equilibrium Points

Point x^* and y^* is called the equilibrium point (zero solution) of model (4) if $x_m = x^*$ and $y_m = y^*$ for all m [19]. Therefore, x^* and y^* is the equilibrium point of model (4) if it satisfies

$$\begin{aligned} x^* &= x^* + \frac{s^n}{n\Gamma(n)}\left(x^*\left(1-\frac{x^*}{\gamma}\right)\right) - \\ &\quad \frac{s^n}{n\Gamma(n)}\left(\frac{(1-c')x^*y^*}{1+\alpha\xi+x^*}\right), \\ y^* &= y^* - \frac{s^n}{n\Gamma(n)}(\delta y^*) \\ &\quad + \frac{s^n}{n\Gamma(n)}\left(\frac{\beta\left((1-c')x^*+\xi\right)y^*}{1+\alpha\xi+x^*}\right). \end{aligned} \qquad (5)$$

Obviously, model (4) has three equilibrium points:

1. The trivial equilibrium point $E_0 = (0,0)$,
2. The axial equilibrium point $E_1 = (\gamma, 0)$, and
3. The co-existing equilibrium point $E_2 = (x^*, y^*)$ with $x^* = \frac{\delta+\alpha\delta\xi-\beta\xi}{\beta(1-c')-\delta}$ and $y^* = \left(1-\frac{x^*}{\gamma}\right)\left(\frac{1+\alpha\xi+x^*}{1-c'}\right)$.

Numerical Method and Simulations

To illustrate the model (4), some results of our numerical simulations are provided. The numerical simulations are performed using parameter values as in Table 1.

Table 1. Parameters Value

Parameters	α	γ	c'	ξ	δ	β	n
Value	0.6	2.6	0.6	0.2	0.08	0.21	0.9

Using parameter values on Table 1, it can be shown that both of equilibrium point $E_0 = (0,0)$ and $E_1 = (2.6, 0)$ exist, but equilibrium point E_2 does not exist.

In Figure 1 to Figure 3, we plot numerical solutions using $s = 0.5, 2.0,$ and 2.1. Equilibrium point E_1 is asymptotically stable while others are unstable. In Figure 1 and Figure 2, a relatively small value of time step (s) is chosen. The solution of prey population is convergent to a constant value, i.e. $x = 2.6$, while the predator population directly goes to extinct. This shows that $E_1 = (2.6, 0)$ is asymptotically stable. For a larger s (that is $s = 2.1$), the numerical solution of prey population is unstable (Fig. 3). Figure 3 show that the prey population oscillates around $x = 2.6$. In more detail, period-2^{nd}-orbits appeares in prey population.

For next simulation, we still use parameters as in Table 1, except $\gamma = 4$, $c' = 0.21$, $\beta = 0.15$, and $\delta = 0.09$. Using those parameters, equilibrium point $E_0 = (0,0)$, $E_1 = (4,0)$, and $E_2 = (2.4842, 1.7288)$ exist.

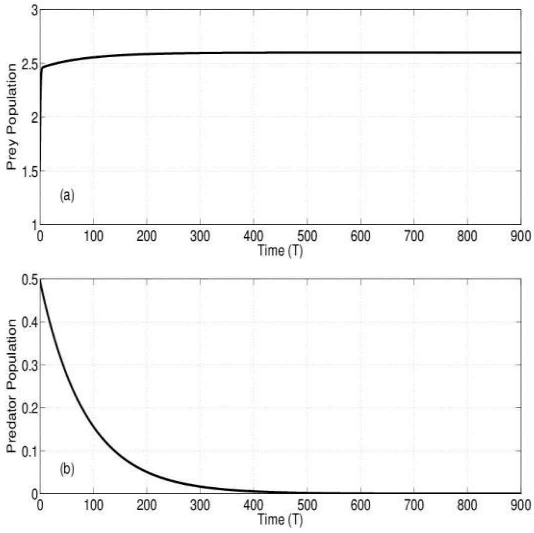

Figure 1. Numerical solution using $\alpha = 0.6, \gamma = 2.6$, $c' = 0.6, \xi = 0.2, \delta = 0.08, \beta = 0.21$, and $n = 0.9$ for $s = 0.5$ on both (a) prey and (b) predator population.

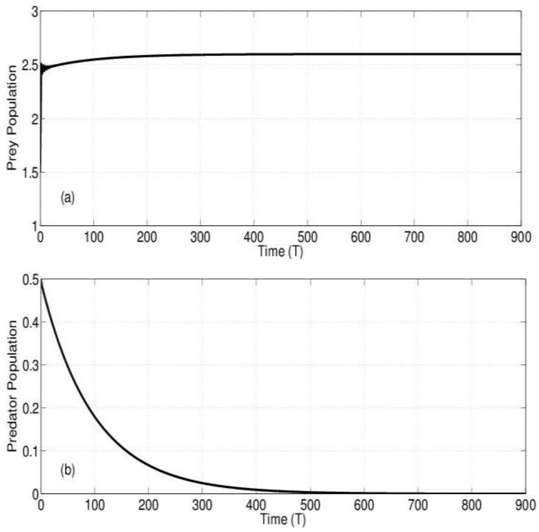

Figure 2. Numerical solution using $\alpha = 0.6, \gamma = 2.6$, $c' = 0.6, \xi = 0.2, \delta = 0.08, \beta = 0.21$, and $n = 0.9$ for $s = 2$ on both (a) prey and (b) predator population.

In Figure 4 and Figure 5, using a relatively small value of s, the numerical solutions of each population are seem to be asymptotically stable and converge to equilibrium point $E_2 = (2.4842, 1.7288)$. In Figure 6, a larger value of s is chosen. In this case, the solution blows up and is not convergent to any point. This indicates that E_2 is unstable with a larger s value.

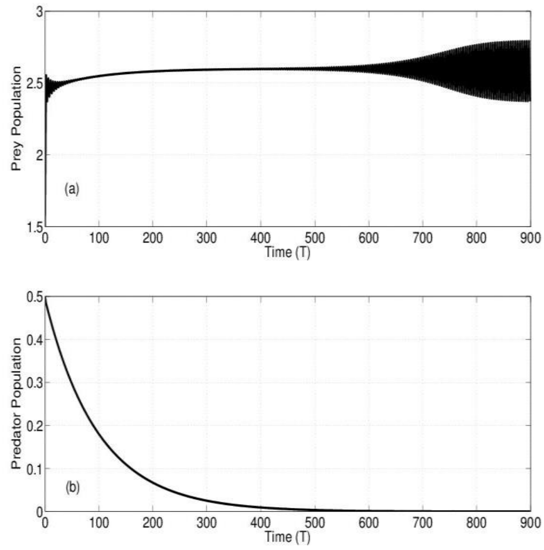

Figure 3. Numerical solution using $\alpha = 0.6$, $\gamma = 2.6$, $c' = 0.6$, $\xi = 0.2$, $\delta = 0.08$, $\beta = 0.21$, and $n = 0.9$ for $s = 2.1$ on both (a) prey and (b) predator population.

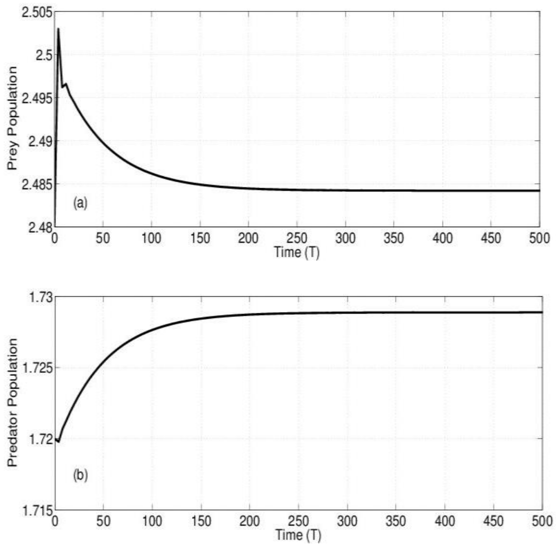

Figure 4. Numerical solution using $\alpha = 0.6$, $\gamma = 4$, $c' = 0.21$, $\xi = 0.2$, $\delta = 0.09$, $\beta = 0.15$, and $n = 0.9$ for $s = 4$ on both (a) prey and (b) predator population.

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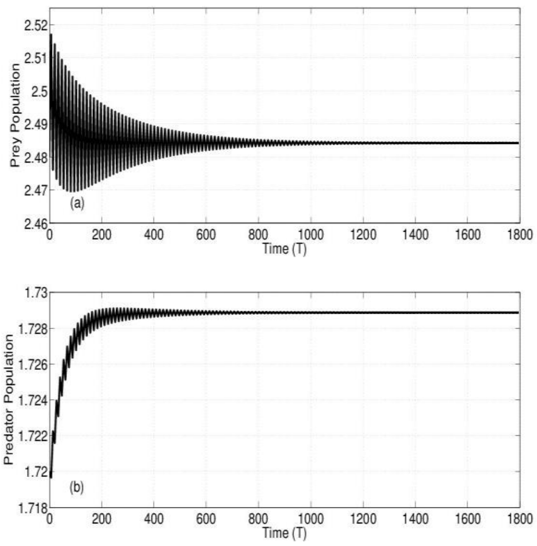

Figure 5. Numerical solution using $\alpha = 0.6$, $\gamma = 4$, $c' = 0.21$, $\xi = 0.2$, $\delta = 0.09$, $\beta = 0.15$, and $n = 0.9$ for $s = 6.8$ on both (a) prey and (b) predator population.

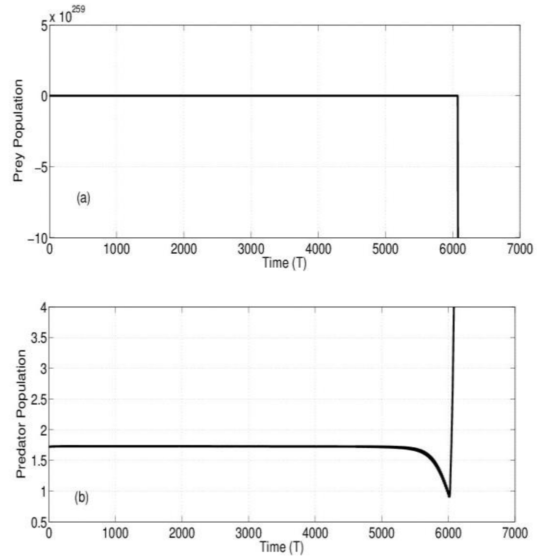

Figure 6. Numerical solution using $\alpha = 0.6$, $\gamma = 4$, $c' = 0.21$, $\xi = 0.2$, $\delta = 0.09$, $\beta = 0.15$, and $n = 0.9$ for $s = 6.91$ on both (a) prey and (b) predator population.

CONCLUSION

In this paper, a discrete fractional-order predator-prey model has been investigated. The discrete model has three equilibrium points, namely the trivial (E_0), the axial (E_1), and the co-existing (E_2) equilibrium point. The numerical simulations show that equilibrium point of discrete model may be stable only for relatively small time step (s). If higher value of s is selected, then complex dynamical behavior such as a period-2^{nd}-orbits appeares. In the future, we will explore the dynamics of the model analytically.

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